Minimum N-Impulse Time-Free Transfers between Elliptic Orbits

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The necessary and sufficient conditions for a maximum of Lawden's Primer Vector as a function of the instantaneous state and adjoints on a conic are obtained for the general time-free case. Complete analytic integrations of the state and adjoint variables during a firing period are derived and incorporated into the maximizing conditions to show that the general switching condition is the smallest (an extension to Lawden's conditions) positive root of a 12th degree polynomial. The paper includes a description of a very efficient program which has been built from these equations to search for the global optima of N-impulse transfers, and illustrates the program's operation by depicting the coplanar extremals.

1. Introduction

THIS paper considers the problem of determining that transfer of a rocket between elliptic orbits which has absolutely minimum Δv; that is, there are no bounds on thrust or on the time allowed for the transfer. This problem is a form of what has become known as Lawden's Problem since his formulation of the governing equations in a series of papers in the 1950's. His fundamental results have been collected in Ref. 1. Lawden showed that when thrust is unbounded, the optimum transfer must consist of combinations of coasting arcs, impulses applied at the maxima of a "Primer Vector" and a special continuous firing which is a singular arc in his formulation. The particular problem of optimal time-free transfer between ellipses was greatly simplified in the middle 1960's by the discovery† that the continuous firing mode was not optimal. The relative simplicity of the analysis of impulsive firings on ellipses and the clearly practical application to many high thrust missions has subsequently attracted many authors[4]; nevertheless, the central problem, that of maximizing the Primer Vector, has been solved exactly only in very special cases. Therefore, although many interesting results have come from investigations of approximations to the Primer Vector,[4] and many exact inequalities limiting the range of the optimum firing angles and multi-impulse transfers have been derived,[5,6] the precise generation of extremals has remained a problem in which necessary conditions are met on a digital computer.

There are two basic types of programs now available to generate minimum Δv orbit transfers. One type minimizes the total Δv expressed as a function of the $3 + 4(N - 2)$ independent parameters of the $N, (N \geq 2)$, impulses. It can be shown‡ that when the parameters are taken to be suitable orbit elements on the coasting arcs, then the $3 + 4(N - 2)$ equations which render Δv stationary are precisely the conditions of Lawden corresponding to a stationary Primer Vector. Therefore, when the solutions of those equations are reduced by testing the Primer Vector magnitude along the coasting arcs, they potentially yield those extremals which satisfy Lawden's conditions. The program of Eckels,[8] the program of McCue, Bender and Lee[9] and that program developed by Lion, Handelsman, Jezewski and Rozendaal[10] are basically of this parameter-optimization type, although they differ widely in the parameterization particularly suited to individual investigations. It has been demonstrated that such programs can yield globally optimum transfers with reasonable computation time if the minimum number of free impulses is assumed. (Reference 9 describes a very satisfactory method; it uses a rough contour map plotting locally minimum Δv for each pair of departure and arrival angles coupled to a steepest-descent into each of the most promising valleys of the contour map.) However, the addition of four parameters for each additional impulse seems to eliminate these parameter-optimization methods from the study of more general transfers between large families of ellipses.

When several impulses are allowed and all optimum transfers between families of orbits are of interest, then a much more promising approach is that of generating the field of extremals in an orbit element state space by starting with an initial state and some associated adjoint parameters and integrating along the state and adjoints while maximizing the instantaneous Hamiltonian. By varying the adjoints over all possible values, and ending each extremal only when it intersects another having lower cost (which may be after many impulses), eventually the state space is filled with globally optimum N-impulse trajectories from an initial orbit to every other orbit. Of course, the computational practicality of this method depends on the complexity of the answer (that is, on the number of extremals required to delineate the important surfaces in state space), but that is equally true of any numerical method. The basic advantage to this method is that the program does not iterate through a series of non-optimal trajectories in order to arrive at one extremal, but generates only extremals. The primary purpose of this paper is to present the analysis involved in formulating a very efficient version of such a program.

A practical demonstration of the use of such a program in solving an orbit transfer problem was made first by Breakwell,[11] who computed a family of coplanar extremels, and by Moyer[12] and Winn,[13,14] who presented the solutions to the coapsidal "Generalized Hohmann Transfers." Winn's program was particularly efficient in that he was able to make use of his knowledge that the extremal was composed of horizontal coapsidal firings to integrate analytically the state and adjoint variables. This also allowed him to reduce the iterative testing of the primer vector magnitude along each instantaneous coasting arc to a single iterative test at the opposite apse. The program described below operates in a

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† Some further discussion and references for the coplanar proof are available in Ref. 2. The extension to three dimensions is given in Ref. 3.

‡ The details of the correspondence in the coplanar, two-impulse case are given by Lawden in Ref. 7. There is no doubt that there are very similar relations in the three dimensional, N-impulse case.

very similar way; it essentially generalizes to arbitrary ellipses that program used by Winn.

2. Maximizing the Hamiltonian

Lawden's statement of the necessary conditions for the extremals of minimum-fuel, time-free orbit transfer in a central force field can be derived directly by taking the orbit's elements as state variables and applying the Maximum Principle. When the independent variable is chosen proportional to the characteristic velocity, the rates of change of state variables can be written

$$x_i' = \bar{d}_i(x_i,u)\cdot\bar{\beta} \qquad i,j = 1,2\ldots 5 \tag{1}$$

in which $\bar{\beta}$, the unit vector in the direction of thrust and u an angular variable in the instantaneous orbit plane, are controls, and the problem is to generate the extremals which minimize the independent variable between end states. The variational Hamiltonian, $H = \lambda_j\bar{d}_j\cdot\bar{\beta}$, is constant and may be taken equal to unity. Then the optimal $\bar{\beta}$ is $\lambda_j\bar{d}_j$, which hereafter will be referred to as $\bar{\lambda}_V^*$,§ and the optimal u is that which maximizes $[\bar{\lambda}_V^*(u)]^2$.

Using any (independent) set of orbit elements, the expression for $\bar{\lambda}_V^*$ is found to have the form¶

$$\bar{\lambda}_V^* = \Lambda_1 \sin(u + \Lambda_2)\bar{e}_R + [\Lambda_3 + (1+\psi)\Lambda_1 \times \cos(u+\Lambda_2)]/\psi\bar{e}_L + \Lambda_4\sin(u+\Lambda_5)/\psi\bar{e}_h \tag{2}$$

in which $\bar{e}_R,\bar{e}_L$, and $\bar{e}_h$ are the unit vectors in the radial, circumferential and out-of-plane directions, and $\psi \equiv 1 + \epsilon\cos f$, where ϵ is the eccentricity and f is $u - \omega$, the true anomaly. The Λ's are functions of the x_j and λ_j. Their functional forms are easily obtained for any particular set of x_j from the $\bar{d}_j$ given in orbital mechanics books, but it will be seen below that it is sufficient to know the general form of u's appearance. Since any set of x_j and λ_j are constant during a coast, the form given can as well be obtained directly from Lawden's general integration for $\bar{\lambda}_V$ on a coasting arc.

The Hamiltonian will be maximized at some u if $[\bar{\lambda}_V^*(u)]^2 - [\bar{\lambda}_V^*(u+\Delta u)]^2 \geq 0$ for all Δu, with equality only at $\Delta u = 2\pi,4\pi\ldots$. Defining $t \equiv \tan(\Delta u/2)$ so that, for example,

$$\Lambda_1\sin(u+\Delta u+\Lambda_2) = (1-t^2)/(1+t^2)\Lambda_1\sin(u+\Lambda_2) + 2t/(1+t^2)\Lambda_1\cos(u+\Lambda_2) \tag{3}$$

then this inequality becomes the ratio of two sixth-degree polynomials in t whose coefficients are functions of the parameters appearing in Eq. (2). The denominator is proportional to $[1+\epsilon\cos(f+\Delta u)]^2$ and so necessarily positive. The term in the numerator which is not multiplied by t vanishes identically (by definition of t) so the parameters in Eq. (2) must be chosen to make the coefficient of the linear term equal to zero, or else the inequality will never be satisfied for some sufficiently small t. This, of course, corresponds to the usual $H_u = 0$ condition. The maximizing test then has the form:

$$t^2[\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4] \geq 0 \text{ for all } t$$

The general test for a positive definite quartic is developed in Appendix A. Using the abbreviations $z_1 \equiv 4\alpha_0\alpha_4 - \alpha_1\alpha_3$ and $z_2 \equiv \alpha_1^2\alpha_4 + \alpha_0\alpha_3^2 - \alpha_1\alpha_2\alpha_3$, it is found there that the requirements are

$$\alpha_0 > 0 \tag{4a}$$

$$16\alpha_0^2(\alpha_2^2 + 3z_1) \geq (3\alpha_1^2 - 8\alpha_0\alpha_2)^2 \text{ if } 3\alpha_1^2 > 8\alpha_0\alpha_2 \tag{4b}$$

$$z_1[(\alpha_2^2 - z_1)^2 + 9\alpha_2 z_2] - \alpha_2^3 z_2 - 27z_2^2/4 > 0 \tag{4c}$$

and that (4c) will be violated first as the coefficients change during a firing, requiring a switch through an angle described by

$$t_s = \frac{2(\alpha_1^2 - 4\alpha_0\alpha_2)(\alpha_2^2 + 3z_1) - 4\alpha_0(9z_2 - 8\alpha_2 z_1)}{4\alpha_0\alpha_3(\alpha_2^2 + 3z_1) + \alpha_1(9z_2 - 8\alpha_2 z_1)} \tag{5}$$

It is shown in Appendix B that this $\alpha_0 > 0$ case generates an impulse. When $\alpha_0 = 0$, the maximizing conditions are simply: $\alpha_1 = 0$, $\alpha_2 \geq 0$, $\alpha_4 \geq 0$ and $\alpha_3^2 \leq 4\alpha_2\alpha_4$. It is shown in Appendix B that this generates either the nonoptimal** singular arc of the $\bar{R}$, $\bar{V}$ formulation, or generates a family of impulsive extremals included in those generated by parameters satisfying Eq. (4); hence, $\alpha_0 = 0$ need not be considered further.

The $H_u = 0$ condition and the α_i are explicit functions of the eight variables in Eq. (2), but because $(\bar{\lambda}_V^*)^2$ can be scaled equal to 1, and u just depends on an arbitrary reference axis, it turns out that they can be written as functions of six independent parameters. Then the switching condition is basically a surface in five dimensions. The choice of parameters is quite free, but it seems most informative to include among them the four variables which describe the geometry of the firing. Here these four are chosen to be $\epsilon\sin f$, ψ, and the firing angles ϕ and Υ defined by $\bar{\lambda}_V^* = \sin\phi\,\bar{e}_R + \cos\phi\cos\Upsilon\,\bar{e}_L + \cos\phi\sin\Upsilon\,\bar{e}_h$. The two which remain (before an elimination using $H_u = 0$) will be chosen from dynamic considerations which will become apparent below. They are defined by

$$k \equiv [\Lambda_3 + \Lambda_1\cos(u+\Lambda_2)]/(\psi\cos\phi)$$

$$j \equiv \{\Lambda_4\sin(u+\Lambda_5)\sin\phi + \Lambda_4\cos(u+\Lambda_5)\cos\phi\cos\Upsilon + [\Lambda_1\cos(u+\Lambda_2)\epsilon\sin f - \Lambda_1\sin(u+\Lambda_2)\epsilon\cos f] \times \cos\phi\sin\Upsilon\}/\cos^2\phi = [\Lambda_4\cos(u+\Lambda_5)/\cos\phi + \epsilon\sin f\sin\Upsilon]\cos\Upsilon + [\tan\phi - k\epsilon\sin f]\sin\Upsilon \tag{6}$$

The $H_u = 0$ condition can be written $k\psi\tan\phi = [\Lambda_4\cos(u+\Lambda_5)/\cos\phi + \epsilon\sin f\sin\Upsilon]\sin\Upsilon - [\tan\phi - k\epsilon\sin f]\cos\Upsilon$. So at any maximum $j\sin\Upsilon - k\psi\tan\phi\cos\Upsilon = \tan\phi - k\epsilon\sin f$, $j\cos\Upsilon + k\psi\tan\phi\sin\Upsilon = \Lambda_4\cos(u+\Lambda_5)/\cos\phi + \epsilon\sin f\sin\Upsilon$. These yield

$$\tan\phi = (j\sin\Upsilon + k\epsilon\sin f)/(1 + k\psi\cos\Upsilon) \tag{7}$$

and an expression for $\Lambda_4\cos(u+\Lambda_5)/\cos\phi$ which can be used to eliminate that combination from the equations for the α_i. Then these α_i, computed from the definition

$$[\bar{\lambda}_V^*(u)]^2 - [\bar{\lambda}_V^*(u+\Delta u)]^2 \equiv \frac{4\cos^2\phi\, t^2(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4)}{(1+t^2)[\psi - 2\epsilon\sin f t + (2-\psi)t^2]^2} \tag{8}$$

are found to be:

$$\alpha_0 = \psi[1 - \psi k^2 - (2 + \psi k^2)\tan^2\phi] - j^2 \tag{9a}$$

$$\alpha_1 = 4\psi\tan\phi(2k - \cos\Upsilon) - 2\epsilon\sin f(1 - 2\tan^2\phi) \tag{9b}$$

$$\alpha_2 = \alpha_0 + \alpha_4 + 8\epsilon\sin f\tan\phi(\cos\Upsilon - k) - 4(\psi - 1) \times [(\cos\Upsilon - k)^2 - \tan^2\phi] \tag{9c}$$

$$\alpha_3 = \alpha_1 + 8(\psi - 1)\tan\phi(\cos\Upsilon - k) + 4\epsilon\sin f[(\cos\Upsilon - k)^2 - \tan^2\phi] \tag{9d}$$

$$\alpha_4 = [(3-\psi)\cos\Upsilon - 2k] \times [2k - \cos\Upsilon] - (\psi - 1)\sin^2\Upsilon \tag{9e}$$

In addition to Eq. (4), many other simple conditions involving these coefficients (such as $\alpha_4 > 0$, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$, $\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 > 0$, $\alpha_0 + \alpha_2 + \alpha_4 > 0$,

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§ $\bar{\lambda}_V^*$ is proportional to the time-free velocity adjoint vector, which is the time-free Primer Vector.

¶ Note that throughout the paper, equations written in the form A/BC will mean $(A/B)C$.

** In the notation of Ref. 15, for example, optimality requires the positivity of W, the fourth derivative of $[\bar{\lambda}_V^*(u)]^2$ on a coasting arc. That corresponds to $\alpha_2 < 0$ when $\alpha_0 = \alpha_1 = 0$, and $\alpha_2 < 0$ does not maximize the Hamiltonian.

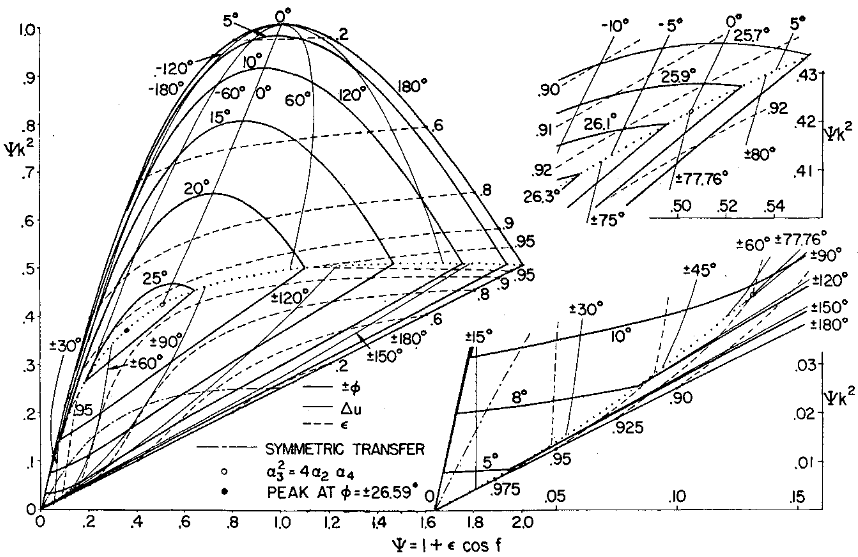

Fig. 1 Coplanar extremals.

and $\alpha_2 > -(4\alpha_0\alpha_4)^{1/2}$) are also necessary in order to make Eq. (8) positive definite. These are all contained in Eq. (4), but the simple forms provide useful constraints on the maximizing parameters and information about the firing mode. For example, $\alpha_0 > 0$ and $\alpha_4 > 0$ bound j,k and $\tan\phi$. Also, since α_4 can be rearranged into $\{(2-\psi)^2 - \psi^2 \sin^2\Upsilon - [(4-\psi)\cos\Upsilon - 4k]^2\}/4$, the $\alpha_4 > 0$ condition with Eq. (7) constrains the optimum to nearly tangential firing near the perigees of highly eccentric ellipses.

3. Comparisons

It is of interest to examine the manner in which the switching conditions presented here limit to the special cases previously discussed in the literature, both as illustrations and to indicate what complexities arise in expanding about those limit cases.

1) $\epsilon = 0$ implies $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_0 + \alpha_4$ so that $z_2 = 0$ and (4c) reduces to $z_1 > 0$. Condition (4b) is always satisfied since $3\alpha_1^2 - 8\alpha_0\alpha_2 = 4\alpha_0\alpha_2 - 3(z_1 + 4\alpha_0^2) < 4\alpha_0(\alpha_2^2 + 3z_1)^{1/2}$. The double maximum occurs when $z_1 = 0$ at $t_s = -2\alpha_0/\alpha_3$. These conditions are also easily derived directly from: $\alpha_0 + \alpha_3 t + (\alpha_0 + \alpha_4)t^2 + \alpha_3 t^3 + \alpha_4 t^4 = (1+t^2)[\alpha_0 + \alpha_3 t + \alpha_4 t^2] > 0$ for all t.

2) The Generalized Hohmann Transfer (GHT), for which $\epsilon \sin f = j = 0$, implies $\tan\phi = \alpha_1 = \alpha_3 = z_2 = 0$, so Eq. (4c) reduces to $\alpha_4 > 0$ and Eq. (4b) reduces to $4\alpha_0\alpha_4 \geq \alpha_2^2$ if $\alpha_2 < 0$. Again, these same conditions are easily derived directly from: $\alpha_0 + \alpha_2 t^2 + \alpha_4 t^4 > 0$ for all t. However, note that in any GHT the α_2 can be rearranged into $\psi^2 \sin^2\Upsilon + (1+\psi)\alpha_4 + (4-\psi^2)(\cos\Upsilon - k)^2$. This is always positive while α_4 is positive, so the (4b) condition is not restrictive and the double maximum always occurs when $\alpha_4 = 0$ at $\Delta u = 180°$. It is seen from Eq. (9) that the particular simplicity of the coplanar GHT is represented here by the factoring of $\alpha_4 = 0$ when $\sin\Upsilon = 0$.

3) The coplanar transfer, for which $\sin\Upsilon = j = 0$, entails no generically zero coefficients and so requires full utilization of (4). However, these transfers do have two simplifying characteristics. First, it happens that the firing angles are rather tightly constrained between the velocity vector and the local horizontal[5,16] since $\alpha_4 > 0$ requires $1 \leq 2k/\cos\Upsilon \leq 3 - \psi$ so Eq. (7) gives $\psi/(2+\psi) \leq \psi \tan\phi \cos\Upsilon/(\epsilon \sin f) \leq \psi(3-\psi)/[2 + \psi(3-\psi)]$. This has allowed considerable progress in approximating the transfers with expansions considering nearly-tangent and nearly-horizontal firings.[4] Second, the problem involves only three parameters, so that even the most impenetrable functions are readily examined with simple plots. Several previous studies[16–19] have made use of these dimensional advantages by presenting interesting geometrical constructions and plots of many of the switching conditions, but none included the interesting details of the triple-maximum regions beginning near $\epsilon = .925$. The general case can be simply illustrated by plotting Eq. (4) as a three-dimensional volume (the inside of the wedge in Fig. 1) and using Eqs. (5) and (7) to compute the Δu and ϵ associated with each surface point. It will be seen below that Fig. 1 also represents a complete description of all coplanar extremals.

4. Evolution of the Parameters

The independent variable is chosen to be the dimensionless quantity $\tau \equiv h^*\Delta v/\mu$, in which h^* is a reference value of angular momentum, Δv is the characteristic velocity added and μ is the gravitational constant of the central body. Defining $L \equiv \log(h/h^*)$, where h is $|\bar{R} \times \bar{V}|$, then $e^L \sin\Upsilon$, ψe^{-2L} and ϕ are constant while $e^L \cos\Upsilon$ and $e^{-L}\epsilon \sin f$ increase at the (constant) rates $(e^{2L}/\psi)\cos\phi$ and $\sin\phi$. The evolution of the two remaining parameters in Eq. (9) can be determined from the (constant) radial components of $\bar{\lambda}_R{}^*$ and $\bar{\lambda}_V{}^* \times \bar{\lambda}_R{}^*$,†† the latter being equivalent to the dot product of $\bar{\lambda}_V{}^*$ with the constant vector $\bar{R} \times \bar{\lambda}_R{}^* + \bar{V} \times \bar{\lambda}_V{}^*$.[20] It is shown in Appendix B that $\bar{\lambda}_R{}^* = -h/R^2(\bar{\lambda}_V{}^*)_u$, where subscript u means partial with respect to u, so

$$\begin{aligned} hR/\mu(\bar{\lambda}_R{}^*) &= [\Lambda_3 + \Lambda_1 \cos(u+\Lambda_2)]\bar{e}_R + \\ &[\Lambda_1 \sin(u+\Lambda_2) - \{\Lambda_3 + \Lambda_1\cos(u+\Lambda_2)\}\epsilon \sin f/\psi]\bar{e}_L - \\ &[\Lambda_4 \cos(u+\Lambda_5) + \Lambda_4 \sin(u+\Lambda_5)\epsilon\sin f/\psi]\bar{e}_h \\ h/\mu(\bar{R} \times \bar{\lambda}_R{}^* + \bar{V} \times \bar{\lambda}_V{}^*) &= \Lambda_4 \sin(u+\Lambda_5)\bar{e}_R + \\ &\Lambda_4\cos(u+\Lambda_5)\bar{e}_L + [\Lambda_1\cos(u+\Lambda_2)\epsilon\sin f - \\ &\Lambda_1 \sin(u+\Lambda_2)\epsilon\cos f]\bar{e}_h \end{aligned} \tag{10}$$

and therefore both $e^{-L}j$ and $e^L k$ are constant.

With these relations, it is easily seen that the set of coefficients $(\alpha_4\psi, \alpha_3\psi^{1/2}, \alpha_2, \alpha_1\psi^{-1/2}$ and $\alpha_0\psi^{-1})$, which will be called $\alpha_i{}^*$, are polynomials of degree i in τ, so Eq. (4c) would be a 12th-degree polynomial if it were explicitly written out as a function of input parameters and τ. For "neighboring transfers," in which $\Delta\tau$ is small, it may be of interest to obtain the first few coefficients analytically to obtain an analytic solution for the value of τ at the double maximum, but usually it is more efficient to compute Eq. (4c) as a function of $\alpha_i{}^*$. The degree of Eq. (4c) is of interest, however, in indicating the many values of τ which could satisfy the double maximum

†† With τ as independent variable, $\bar{\lambda}_V{}^* = \mu/h^*\,\bar{\lambda}_V(u,x_j,0,\lambda_j)$ and $\bar{\lambda}_R{}^* = \mu/h^*\bar{\lambda}_R(u,x_j,0,\lambda_j)$.

condition. Only the lowest can be used in generating a locally optimal series of impulses, which supplements the usual description of a switch point as a "double-maximum of the Hamiltonian." The geometric significance of the distinction is that other double-maxima may well occur after some Hamiltonian peak has risen past 1 and then subsided.

5. A Program for Generating Extremals

In conclusion, the relations previously developed will be correlated by describing the operation of the program which has been developed to generate time-free extremals.

1) An initial set of parameters $(\epsilon, f, \Upsilon, j\psi^{-1/2}, k\psi^{1/2})_0$ which satisfy Eq. (4) is input. There is the possibility of some fumbling in picking initially maximizing inputs, but often some known analytical approximations[4] can be used to specify regions of interest. In case nothing at all is known about the problem studied, then it is useful (and informative) to sketch roughly, for several j and Υ other than zero, the figures similar to Fig. 1.

The initial e^L is arbitrary, just scaling τ without affecting the extremal. It is usually convenient to input it equal to 1 so that τ is the fractional velocity increment $h_0\Delta v/\mu$.

2) τ is incremented by a succession of $\Delta\tau$. The corresponding values of α_i^* and (4c) are computed until (4c) is zero at some τ_s.

3) The values of $\Lambda_1 \sin(u + \Lambda_2)$, $\Lambda_1 \cos(u + \Lambda_2)$, Λ_3, $\Lambda_4 \sin(u + \Lambda_5)$ and $\Lambda_4 \cos(u + \Lambda_5)$ are computed from the known values of $\epsilon, f, \Upsilon, j, k$ at τ_s, and all variables are switched to the new peak by adding the Δu_s computed from Eq. (5); then the next impulse continues by incrementing τ as in 2).

The end result is a series of known impulses and coasting arcs leaving from a point f on an initial orbit ϵ_0. The description of the extremal path in state-space is obtained by computing $\bar{h} = \bar{R} \times \bar{V}$ and $\bar{\epsilon} = \bar{V} \times \bar{h}/\mu - \bar{e}_R$ whenever desired while incrementing τ. A region of state-space is investigated by varying the inputs over a range of values and continuing each extremal until it intersects another of lower cost or until it exceeds the instantaneous bi-parabolic transfer cost: $\tau_{BP} = e^{-L_0}\{[2(1 + \epsilon_0)]^{1/2} - 1 - \epsilon_0\} + e^{-L}\{[2(1 + \epsilon)]^{1/2} - 1 - \epsilon\}$. The computation of each extremal (including several impulses and several state outputs during each impulse) is only a small fraction of a second on an IBM 360-67 computer.

This program has been used repeatedly during the past two years both to supplement analytical developments by examining the effects of higher order terms[21] and to examine transfer modes in multi-impulse regions of state-space. The presentation of any complete state-space mapping is beyond the scope of this paper, but Fig. 1 can be interpreted as a (coplanar) illustration of the extremal families generated by the program described here. There ψ is monotonically increasing or decreasing with τ while ψk^2 and ϕ are constant, so the bundle of horizontal lines inside the wedge represent the coplanar extremals corresponding to any one impulse. A line ending at the surface point P having coordinates $(\epsilon^*, \Delta u^*)$ will "switch" to that point $P'(\epsilon^*, -\Delta u^*)$ which does not require crossing a dotted "triple maximum" ridge; then it continues horizontally (into the wedge's interior) with the ψk^2 and ϕ of P'. Continuing in this manner from any initial interior point, one can follow by eye on Fig. 1 the histories of all coplanar impulse-coating series as computed by the extremal program.

Appendix A: The Positive Quartic

The maximizing condition may be written in the form:

$$\alpha_0 t^{-4} + \alpha_1 t^{-3} + \alpha_2 t^{-2} + \alpha_3 t^{-1} + \alpha_4 > 0 \text{ for all } t \quad \text{(A1)}$$

This requires $\alpha_0 > 0$, in order to make the quartic positive for small t. Dividing through by α_0, and introducing the new independent variable $y \equiv \alpha_1/(4\alpha_0) + t^{-1}$ to eliminate the

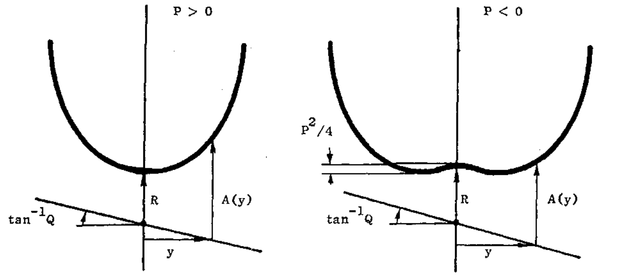

Fig. 2 Possible forms for $A(y)$.

cubic term, (A1) becomes

$$A(y) \equiv y^4 + Py^2 + Qy + R > 0 \text{ for all } y \quad \text{(A2)}$$

in which

$$P = (8\alpha_2\alpha_0 - 3\alpha_1^2)/(8\alpha_0^2)$$
$$Q = [\alpha_1(\alpha_1^2 - 4\alpha_2\alpha_0) + 8\alpha_3\alpha_0^2]/(8\alpha_0^3) \quad \text{(A3)}$$
$$P^2 + 12R = (\alpha_2^2 + 12\alpha_4\alpha_0 - 3\alpha_1\alpha_3)/\alpha_0^2$$

The possible forms of $A(y)$ are sketched in Fig. 2. Defining $b = P(P^2 + 12R)^{-1/2}$, it is evident from the figure that the necessary and sufficient conditions for $A(y)$ to be positive definite are that b be between $-1/2$ and 1 and that Q^2 be less than its tangency value. A tangency can only occur at some y_s satisfying

$$y_s^4 + Py_s^2 + Qy_s + R = 0$$
$$4y_s^3 + 2Py_s + Q = 0 \quad \text{(A4)}$$

These yield the real root

$$y_s = -\text{sgn}Q\{[-P + (P^2 + 12R)^{1/2}]/6\}^{1/2} \quad \text{(A5)}$$

and the tangency value: $(Q^2)_T = 2/27[1 + 3b - 4b^3](P^2 + 12R)^{3/2}$. Since this expression goes negative for $b > 1$, the conditions for a positive definite $A(y)$ can be written simply: $1 + 2b > 0$ and $Q^2 < (Q^2)_T$. The double-maximum of the Hamiltonian occurs when $Q^2 = (Q^2)_T$ and the location of the second maximum is then given by (A5). The side condition $1 + 2b > 0$ then precludes a third maximum higher than two equal relative maxima. [The case of three equal maxima is simply $b = -\frac{1}{2}$ which is $Q = 0$, $P = -(4R)^{1/2}$, and the locations are $y_s = \pm(-P/2)^{1/2}$.]

For both analytical and numerical work there are some advantages in eliminating the fractional powers and in expressing the test conditions directly as functions of the α_i. The tangential inequality is $S > 0$ when

$$S \equiv 1 + 3b - 4b^3 - 27Q^2/2(P^2 + 12R)^{-3/2} \quad \text{(A6)}$$

or

$$27Q^2/2 + 4P^3 - 3P(P^2 + 12R) < (P^2 + 12R)^{3/2} \quad \text{(A7)}$$

But since $S \leq 2$ when $b \geq -\frac{1}{2}$, with a grazing equality only when $Q = 0$ and $b = \frac{1}{2}$, the $S > 0$ condition could as well be written $S(2 - S) > 0$, and that latter form amounts to squaring both sides of (A7). When using the squared form, however, it must be kept in mind that equality represents either a double maximum or the $Q = 0$, $b = \frac{1}{2}$ point. (In practice, the program described herein switches to the new maximum only when $S(2 - S)$ goes infinitesimally negative, which eliminates any ambiguity.) Substituting (A3) into (A7) yields

$$3(9z_2 - 8\alpha_2 z_1) + 2\alpha_2(\alpha_2^2 + 3z_1) < 2(\alpha_2^2 + 3z_1)^{3/2} \quad \text{(A8)}$$

in which $z_1 \equiv 4\alpha_0\alpha_4 - \alpha_1\alpha_3$ and $z_2 \equiv \alpha_1^2\alpha_4 + \alpha_0\alpha_3^2 - \alpha_1\alpha_2\alpha_3$. The squared form is given in Eq. (4c).

The side condition $2b + 1 > 0$, or $2P + (P^2 + 12R)^{1/2} > 0$, is automatically satisfied if $P > 0$, so its squared form need be satisfied only if $P < 0$ as stated in (4b).

The radicals can be eliminated from the computation of y_s by using (A4) directly. This gives

$$y_s = -Q/(2P + 4y_s^2) = -3Q/[4P + 2(P^2 + 12R)^{1/2}]$$

$$t_s^{-1} = y_s - \frac{\alpha_1}{4\alpha_0} = \frac{\alpha_1\alpha_2 - 6\alpha_0\alpha_3 - \alpha_1(\alpha_2^2 + 3z_1)^{1/2}}{8\alpha_0\alpha_2 - 3\alpha_1^2 + 4\alpha_0(\alpha_2^2 + 3z_1)^{1/2}} \quad \text{(A9)}$$

and the form in Eq. (5) is obtained by eliminating the radical with the equality in Eq. (A8). The price paid for this form is the indeterminacy at a triple maximum. In practice, however, the numerical (double precision) error is negligible when investigating the extremal field near a triple maximum, and for analytical work at a triple maximum there are much simpler forms for all these equations; viz. 1) If $\alpha_1 = 0$, then $Q = 0$ requires $\alpha_3 = 0$ and $b = -1/2$ requires $\alpha_2 < 0$ and $\alpha_2^2 = 4\alpha_0\alpha_4$. The locations, from (A5), are $t_s^{-1} = \pm[-\alpha_2/(2\alpha_0)]^{1/2}$. 2) If $\alpha_1 \neq 0$, then the triple maximum conditions are $\alpha_1(\alpha_1^2 - 4\alpha_0\alpha_2) + 8\alpha_0^2\alpha_3 = 0$, $8\alpha_0\alpha_2 < 3\alpha_1^2$ and $\alpha_1^2\alpha_4 = \alpha_0\alpha_3^2$, which is obtained easily after noticing that (4b) can be rearranged into the alternate form: $\alpha_1^2\alpha_4 - \alpha_0\alpha_3^2 \geq Q[\alpha_1(3\alpha_1^2 - 4\alpha_0\alpha_2) - 8\alpha_0^2\alpha_3]/8$. The locations are $t_s^{-1} = [-\alpha_1 \pm (3\alpha_1^2 - 8\alpha_0\alpha_2)^{1/2}]/(4\alpha_0)$.

Appendix B: Optimal Maneuvers

The analysis in Sec. 2 did not assume any particular firing mode. The x_j' and corresponding λ_j' $(= -H_{x_j})$ are functions of x_j,λ_j and some u which is determined by the condition that it continuously maximize $(\bar{\lambda}_V{}^*)^2$ by maintaining $(\bar{\lambda}_V{}^*)^2{}_u \equiv 0$. As long as $(\bar{\lambda}_V{}^*)^2{}_{uu} \neq 0$, u's rate of change can be computed from $0 \equiv (\bar{\lambda}_V{}^*)^2{}_{uu}u' + (\bar{\lambda}_V{}^*)^2{}_{ux_j}x_j' + (\bar{\lambda}_V{}^*)^2{}_{u\lambda_j}\lambda_j'$ and then that value can be substituted into $\bar{R}' = \bar{R}_u u' + \bar{R}_{x_j}x_j'$ and $\bar{\beta}' = \bar{\beta}_u u' + \bar{\beta}_{x_j}x_j' + \bar{\beta}_{\lambda_j}\lambda_j'$ to describe the firing mode. If it should turn out that $\bar{R}' = \bar{\beta}' = 0$, then the optimum firing mode is impulsive, otherwise not.

The algebra of this computation can be greatly simplified if it is carried out after examining some general relationships between orbit elements and $\bar{R},\bar{V}$ coordinates. When nQ's, nP's, nq's and np's are related by the definitions $q_j \equiv q_j(Q_k)$ and $p_j \equiv P_k(\partial Q_k/\partial q_j)$, then these relations are a form of contact transformation, and for that it is well known that the value of the Poisson brackets of any two functions (say, F and G) is invariant:

$$\{F,G\} = \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} = \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k}$$

It follows immediately that

$$\{P_i,P_j\} = \{Q_i,Q_j\} = 0$$

$$\{Q_i,P_j\} = -\{P_i,Q_j\} = \delta_{ij} \quad \text{(B1)}$$

The transformation from $(\bar{R},\bar{V},\bar{\lambda}_R,\bar{\lambda}_V)$ to $(u,x_j,\lambda_u,\lambda_j)$ is just such a transformation, since the adjoints in an optimization problem represent the payoff sensitivities so that when $P_k = \partial\$/\partial Q_k$, then $p_j = \partial\$/\partial q_j = P_k(\partial Q_k/\partial q_j)$. In addition, when the (P,Q) are canonically associated with a Hamiltonian $P_k\dot{Q}_k$ in an optimization problem, then the Hamiltonian in the transformed problem is $p_j\dot{q}_j = p_j(\partial q_j/\partial Q_k)\dot{Q}_k$, which is just $P_k\dot{Q}_k$ expressed as a function of q and p.

The time-unconstrained, unbounded thrust problem specifically considered in this paper is distinguished by $\lambda_u \equiv 0$, but λ_u must be explicitly considered in Poisson bracket operations on $\bar{\lambda}_R(u,x_j,\lambda_u,\lambda_j)$ and $\bar{\lambda}_V(u,x_j,\lambda_u,\lambda_j)$ before evaluating the results at $\lambda_u = 0$. In the problem considered here, the rate of change of an arbitrary function F is $F_u u' + F_{x_j}(\bar{\lambda}_V{}^*\cdot\bar{\beta})_{\lambda_j} - F_{\lambda_j}(\bar{\lambda}_V{}^*\cdot\bar{\beta})_{x_j}$. Because $(\bar{\lambda}_V{}^*\cdot\bar{\beta})_{\lambda_u} = (\bar{\lambda}_V{}^*\cdot\bar{\beta})_u = 0$, this can as well be written $F' = F_u u' + \{F,\bar{\lambda}_V{}^*\cdot\bar{\beta}\}$ and any λ_u in F can be set equal to zero either before or after the computation without affecting the result. The relation between $\bar{\lambda}_V$ and $\bar{\lambda}_V{}^*$ can be computed by solving for $\bar{\lambda}_V$ from the $P_k = p_j(\partial q_j/\partial Q_k)$ relations, but it is found more easily by comparing the usual time-formulated Hamiltonians: $\bar{\lambda}_R\cdot\bar{V} + \bar{\lambda}_V\cdot(\bar{g} + \bar{a}) = \lambda_u(h/R^2 - \dot{\Omega}\cos i) + \lambda_j\bar{d}_j\cdot\bar{a}h^*/\mu$, where $h = |\bar{R} \times \bar{V}|$, Ω and i are nodal and inclination angles, $\bar{g}$ is inverse-square acceleration and $\bar{a}$ is the firing acceleration. Since $\lambda_u = \bar{\lambda}_R\cdot(\bar{R})_u + \bar{\lambda}_V\cdot(\bar{V})_u = (\bar{\lambda}_R\cdot\bar{V} + \bar{\lambda}_V\cdot\bar{g})R^2/h$, then $\bar{\lambda}_V{}^*\cdot\bar{\beta} = \bar{\lambda}_V\cdot\bar{\beta}\mu/h^* + \lambda_u\Omega'\cos i$. Also, $F_u = \{F,\lambda_u\} = \{F,\bar{\lambda}_R\cdot\bar{V} - \bar{\lambda}_V\cdot\bar{R}\mu/R^3\}R^2/h \equiv F_T R^2/h$ so, finally, $F' = F_T R^2/h(u' + \Omega'\cos i) + \mu/h^*\{F,\bar{\lambda}_V\cdot\bar{\beta}\}$. [Technically, F_T is just a symbol given to the Poisson bracket operation on F, but the notation is meant to call attention to the fact that F_T equals that portion of F's time derivative which does not contain a. For example: $(\bar{\lambda}_{Tv}{}^*)_u = (\bar{\lambda}_V{}^*)_T R^2/h = -\bar{\lambda}_R{}^*R^2/h$.] Now the computations described in the first paragraph of this appendix are easily carried out, and we obtain:

$$\bar{R}' = \bar{V}R^2/h\,(u' + \Omega'\cos i) \quad \text{(B2)}$$

$$\bar{\beta}' = (\bar{\lambda}_V{}^*)_T R^2/h(u' + \Omega'\cos i) \quad \text{(B3)}$$

$$[(\bar{\lambda}_V{}^*)_T{}^2]' = (\bar{\lambda}_V{}^*)_{TT}{}^2 R^2/h(u' + \Omega'\cos i) = 0 \quad \text{(B4)}$$

$$[(\bar{\lambda}_V{}^*)_{TT}{}^2]' = (\bar{\lambda}_V{}^*)_{3T}{}^2 R^2/h(u' + \Omega'\cos i) \quad \text{(B5)}$$

$$[(\bar{\lambda}_V{}^*)^2{}_{3T}]' = (\bar{\lambda}_V{}^*)_{4T}{}^2 R^2/h(u' + \Omega'\cos i) + 2\mu/h^*[3\mu R^{-4}\sin\phi\,(3\text{–}5\sin^2\phi)] \quad \text{(B6)}$$

It is seen from Eq. (8) that $(\bar{\lambda}_V{}^*)_{TT}{}^2 = -2\alpha_0\cos^2\phi\psi^{-2}h^2R^{-4}$, $(\bar{\lambda}_V{}^*)_{3T}{}^2 = -3\alpha_1\cos^2\phi\psi^{-2}h^3R^{-6}$ and (when $\alpha_0 = \alpha_1 = 0$) that $(\bar{\lambda}_V{}^*)_{4T}{}^2 = -6\alpha_2\cos^2\phi\,\psi^{-2}h^4R^{-8}$. Then either: 1) $\alpha_0 \neq 0$, so Eq. (B4) requires $u' = -\Omega'\cos i$ and the firing is impulsive; 2) $\alpha_0 = 0$, then α_1 must be zero to maximize the Hamiltonian and u' must be determined to maintain (B6) equal to zero. That requires either an impulse using $\sin\phi = 0$ (which generates coplanar, coapsidal extremals when $\alpha_0 = \alpha_1 = 0$ and the Hamiltonian is maximized), or it requires a continuous firing with $\sin\phi \neq 0$ (which generates the nonoptimum singular arc).

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Propagation of Harmonic Waves in Composite Circular Cylindrical Shells. Part II: Numerical Analysis

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The frequency equation for harmonic waves with an arbitrary number of circumferential nodes traveling in composite traction-free circular cylindrical shells established in the first part of this investigation has been programmed for numerical evaluation on an IBM 7094 digital computer. The numerical results obtained are employed in evaluating the effect of the changes of the shell parameters on the frequency and shape of the first few modes of wave propagation. Moreover, the asymptotic limits of the phase velocities for waves having short axial wavelengths are established analytically and verified by the numerical results.

Nomenclature†

R	= radius of the middle surface of the outer layer
a	= inner radius of the shell
d	= outer radius of the shell
b	= outer radius of the inner layer
$h_1 h_2$	= thickness of inner and outer layer, respectively
H	= ratio of the thickness of the inner layer to the thickness of the outer layer
u_x, u_θ, u_r	= axial, tangential and radial components of displacement
$\zeta = h_2/L$	= nondimensionalized wave number in the axial direction
L	= axial half-wave length
n	= number of circumferential waves
ω	= circular frequency
v_{1i}, v_{2i}	= velocity of propagation of dilatational and of shear waves, in an infinite medium, respectively
$\Omega_i = \omega h_2/\pi v_{2i}$	= nondimensionalized frequency
E_i, μ_i	= Young's modulus and shear modulus, respectively
$\rho = \rho_1/\rho_2$	= density ratio
$\mu = \mu_1/\mu_2$	= stiffness ratio
$\alpha_i \beta_i$	= radial wavenumbers, see Eq. (1)
$\bar{\alpha}_i \bar{\beta}_i$	= modulus of α_i and β_i, respectively

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† The subscript i assumes the values 1 or 2, depending on whether reference is made to the inner layer or outer layer, respectively.

Introduction

In the first part of this investigation,[1] the frequency equation has been derived for harmonic waves with an arbitrary number of circumferential nodes traveling in composite traction-free circular cylindrical shells. The composite shells consist of two-concentric circular, cylindrical shells bonded at their interface. For given physical and geometric properties of the shell, the frequency equation constitutes a transcendental relation between the nondimensionalized frequency $\Omega_i = (\omega h_i \rho_i^{1/2}/\pi \mu_i^{1/2})$ $(i = 1,2)$ the nondimensionalized wave number $\zeta = h_2/L$ and the number of circumferential waves n. For any chosen value of n and h_2/L, the frequency equation yields an infinite number of values of Ω_i.

In the second part of this investigation, the frequency equation has been programed for numerical evaluation on an IBM 7094 digital computer. The program computes the value of the determinant at a specified initial value of Ω_2 and at values of Ω_2 incremented by $\Delta\Omega_2$. A root is indicated by a change of sign of the determinant between two successive values $\bar{\Omega}_2$ and $\bar{\Omega}_2 + \Delta\bar{\Omega}_2$. The increment is then halved, and commencing with $\bar{\Omega}_2$, the process is repeated until the root is established to the desired accuracy. Subsequently, starting with a new value of Ω_2 obtained by adding a specified increment to the established root, the program continues to establish any number of successive roots desired. The program can also evaluate the amplitude of the displacement components u_r and u_x for axisymmetric nontorsional motion at a specified number of points across the thickness of the shell.

The numerical results obtained are employed in evaluating the effect of the changes of the shell parameters on the frequency and shape of the first few modes of wave propagation. Moreover, the asymptotic behavior of the frequency lines